# Conic section figure calculations for the Long Trace Profiler

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**Introduction**

A mirror surface is often specified in terms of a conic section to give some ideal imaging relationship between source and image. In 3-space a section of an ellipsoid or a section of one sheet of a hyperboloid might be fabricated. However, if a slab of some material is to be bent into a shape for focusing, a cylindrical ellipse or hyperbola would be made. If the surface is specified to be a segment of any such conic section, then the mirror owner would want to know how far from the ideal conic section the measured surface is. This is accomplished by simply subtracting the measured surface from the ideal surface. Formulas for such ideal surfaces as applied to X-ray mirrors have been presented by Howells [1].

The curvature of bent mirrors is often greater than what conventional interferometers can measure over the aperture of the mirror. In these cases a Long Trace Profiler (LTP) is used to measure the cylindrical figure. This paper documents the way that the software LTPw subtracts a specified conic section from the LTP measurement of the surface to render residual figure or figure error.

**Ellipse geometry**

Suppose that an optical system design requires a source point S, a mirror to focus the rays from source to image, and an image point I. If the mirror surface is a segment of an ellipse, then the source point is at one of the ellipse foci and the image point is at the other focus. By virtue of ellipse geometry, any ray from S to some point on the ellipse surface P will reflect at P and complete its trajectory from P to I.

Typically an optical system design will specify the mirror surface by giving the distance from S to the center of the mirror as r and the distance from the center of the mirror to I as r'. These line segments are also called the pole rays r and r'. In addition, the grazing angle θ (or angle of incidence from normal, 90° - θ) and the length of the mirror L are given. These four parameters (r, r', θ, L) completely specify the elliptical section, as long as they are reasonable values.

However, the usual way of specifying an ellipse is to give the common parameters a (major semiaxis) and b (minor semiaxis). We only need consider the normal form of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (1)$$

with no translation or rotation for now. An invariant of an ellipse is the sum r + r'; i.e., the total distance from S to P and from P to I is the same for P anywhere on the ellipse. From ellipse geometry (Figure 1) we have

$$r + r' = 2a \qquad (2)$$

and

$$b^2 = a^2 - c^2 \, . \qquad (3)$$

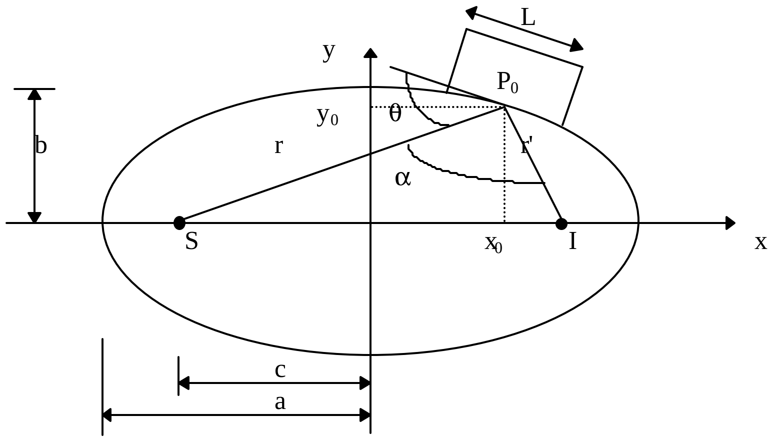


Figure 1. Ellipse geometry.

Thus

$$a = \frac{r + r'}{2} , \tag{4}$$

and (using the Law of Cosines on the triangle with sides r, r’, and 2c)

$$b = \sqrt{\frac{r\ r'}{2}(1 + \cos\alpha)} , \tag{5}$$

describe the ellipse in terms of the common parameters when design parameters r, r', θ are given. Here

$$\alpha = \pi - 2\theta . \tag{6}$$

Let the point on the ellipse that corresponds to the center of the mirror be $P_0 = (x_0, y_0)$. These coordinates can be found by again examining the triangle formed by sides r, r', and 2c. The height of the triangle is $y_0$ and is computed using Heron's formula[2]:

$$y_0 = \pm \frac{r\ r'}{2\,c} \sin\alpha = \pm \frac{r\ r'}{2\sqrt{a^2 - b^2}} \sin\alpha \tag{7}$$

Then $x_0$ is determined from (1) :

$$x_0 = \pm\, a \sqrt{1 - y_0^{\,2} / b^2} . \tag{8}$$

Choose + or - in each of (7) and (8) according to which quadrant describes the situation. LTPw assumes that the surface is pointing upward, so choose quadrant 3 or 4 for a concave surface in which case $y_0$ will be negative.

In case only common parameters are given, r and r' can be found from (4) and (5).

$$r = a \pm \sqrt{a^2 - 2\,b^2 / (1 + \cos\alpha)} ; \tag{9}$$

$$r' = 2\,a - r . \tag{10}$$

**Parabola and hyperbola geometry**

Other conic section forms will be required when the foci have unusual positions. For example, when either the light beam coming from S or going to I is collimated, then one of the foci will be far from $P_0$ compared to the other focus. In this case a parabola should be the conic used, and can be approximated by making either r or r' very large (3e33 using single precision or 9e99 using double precision float numbers).

When the imaging is virtual (either S or I is outside the ellipse), the situation cannot be modelled by making either r or r' negative. In this situation one of the foci has gone beyond the position for collimation and wrapped around to be on the other side (and outside) of the ellipse. The real cartesian representation for this figure is a hyperbola, and is shown in Figure 2. The normal form for a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 . \tag{11}$$

The invariant is expressed by

$$r - r' = 2a , \tag{12}$$

and the "minor semiaxis" is defined as b:

$$b^2 = c^2 - a^2 . \tag{13}$$

In the triangle with sides r, r', and 2c, the angle between r and r' is $\pi - \alpha$. Applying the Law of Cosines to this triangle gives

$$(2c)^2 = r^2 - r'^2 - 2\, r\, r' \cos(\pi - \alpha) . \tag{14}$$

From (13) and trigonometric identities

$$4(a^2 + b^2) = r^2 - r'^2 + 2\, r\, r' \cos\alpha . \tag{15}$$

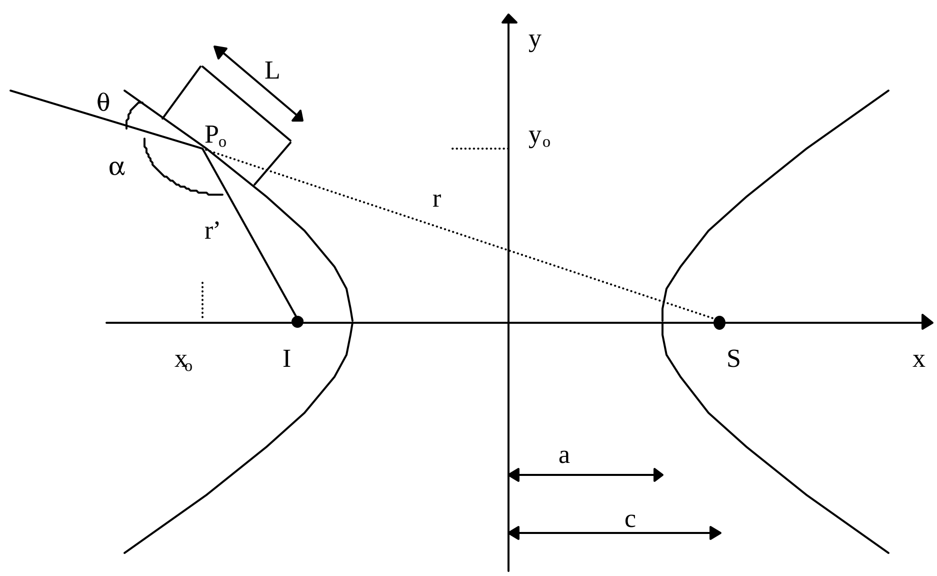


Figure 2. Hyperbola geometry.

Then the common parameters are calculated by

$$b = \sqrt{\frac{r\ r'}{2}(1+\cos\alpha)} \tag{16}$$

and from (12)

$$a = \frac{r - r'}{2} \ . \tag{17}$$

Applying (7) to the triangle in Figure 2 gives

$$y_0 = \pm \frac{r\ r'}{2\,c}\sin(\pi - \alpha) = \pm \frac{r\ r'}{2\sqrt{a^2 + b^2}}\sin\alpha \ , \tag{18}$$

where the included angle $\alpha$ is given by (6). Then $x_0$ is determined from (11) :

$$x_0 = \pm\, a\sqrt{1 + y_0^2 / b^2} \ . \tag{19}$$

Again, design parameters can be found in terms of common parameters. From (16) and (17)

$$r = a \pm \sqrt{a^2 + 2\,b^2 / (1 + \cos\alpha)} \ ; \tag{20}$$

$$r' = r - 2\,a \ . \tag{21}$$

**Expressing the conic section in terms of LTP coordinates**

Although Figures 1 and 2 show the surface as a segment in the upper right or left quadrant of the normal form, the surface could be in any of the four quadrants, depending on the application. The problem of rotating and translating becomes a little easier if the surface is already close to how the data for a surface is presented by the LTP software. So we choose $y_0$ to be negative for a concave surface. The sign of $x_0$ is chosen based on which is longer of r and r'.

The function that calculates the ideal y' of the mirror surface for a given x' in LTP coordinates must be passed five items: the LTP position x', parameters a, b, $\theta$, and the LTP position for the midpoint of the surface $x'_m$. The position x' must be rotated then translated from LTP coordinates to normal coordinates x. The corresponding y value is calculated using (1) or (11), taking the negative value if $y_0$ is negative. Finally y is translated then rotated from normal coordinates to LTP coordinates y' and returned to the calling function.

Thus we need to translate from the x' to x coordinates when the function is called, and we need to translate from the y to y' coordinates when the function returns. The translation from LTP to normal coordinates is given by

$$x = x' - x'_m + x_0 \ , \tag{22}$$

and the translation from normal to LTP coordinates is given by

$$y' = y - y_0 \ . \tag{23}$$

The amount to rotate $\phi$ is obtained from the surface slope at $P_0$. For an ellipse

$$\tan\phi = \left.\frac{dy}{dx}\right|_{x_0} = \frac{b\,x_0}{a\sqrt{a^2 - x_0^2}}\ , \tag{24}$$

and for a hyperbola

$$\tan\phi = \left.\frac{dy}{dx}\right|_{x_0} = \frac{-b\,x_0}{a\sqrt{a^2 + x_0^2}}\ . \tag{25}$$

The direction to rotate depends on which of the quadrants is used. The rotation from normal to LTP coordinates will be the opposite direction as the rotation from LTP to normal coordinates. The difference in sign between an ellipse and a hyperbola shows that for the same quadrant the rotations will be reversed.

**Curvature**

Calculating radius of curvature (which is the reciprocal of curvature: R = 1/C) is done by taking the second derivative of the conic section function. The second derivative should not be with respect to x, because there may be loss of numerical accuracy near slope extremes. Therefore, the ellipse is expressed as a parametric equation.

$$x(t) = a\cos t\ ; \qquad y(t) = b\sin t\ . \tag{26}$$

The radius of curvature for any parametric curve in a plane is [3]

$$R = \frac{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\left|\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dx}{dt}\frac{d^2y}{dt^2}\right|}\ . \tag{27}$$

Thus, taking the first and second derivatives of (26) gives

$$\frac{dx}{dt} = -a\sin t\ ; \qquad \frac{dy}{dt} = b\cos t\ ; \tag{28}$$

$$\frac{d^2x}{dt^2} = -a\cos t\ ; \qquad \frac{d^2y}{dt^2} = -b\sin t\ ; \tag{29}$$

and putting these into (27) gives the radius of curvature for an ellipse:

$$R = \frac{1}{a\ b}\left[a^2\sin^2 t + b^2\cos^2 t\right]^{3/2}\ . \tag{30}$$

The relationship between polar angle $\phi$ and the parameter t is

$$\phi(t) = \arctan\left(\frac{y_o}{x_o}\right) = \arctan\left(\frac{b}{a}\tan t\right) . \tag{31}$$

Therefore,

$$t = \arctan\left(\frac{a\,y_o}{b\,x_o}\right) . \tag{32}$$

The parametric form for a hyperbola is

$$x(t) = a\cosh(t)\,; \qquad y(t) = b\sinh(t) . \tag{33}$$

Taking the first and second derivatives of (33) and putting them into (27) gives the hyperbola's radius of curvature

$$R = \frac{1}{a\,b}\left[a^2\sinh t + b^2\cosh t\right]^{3/2} . \tag{34}$$

Determining the parameter t for a hyperbola is also similar to that of an ellipse.

$$t = \operatorname{arctanh}\left(\frac{a\,y_o}{b\,x_o}\right) , \tag{35}$$

which can be calculated as[2]

$$t = \tfrac{1}{2}\ln\left(\frac{b\,x_o + a\,y_o}{b\,x_o - a\,y_o}\right) . \tag{36}$$

Calculating the radius of curvature at the mirror center is useful in setting the actual elliptical or hyperbolic surface in the first iterations of bending, because it is much faster to first set the surface approximately to a circle and then make small changes to the end couples of the mirror to get the desired conic section shape.

## Acknowledgments

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### References

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3. W. Kaplan, *Advanced Calculus*, (Chapter 1, Problem #7), Addison-Wesley, Reading, MA (1952).